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Computer Simulations for Some One-Dimensional Models of Random Walks in Fluctuating Random Environment

C. Boldrighini,[1] G. Cosimi,[2] S. Frigio,[2] and A. Pellegrinotti[3]

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We report some results of computer simulations for two models of random walks in random environment (rwre) on the one-dimensional lattice $\mathbb{Z}$ for fixed space–time configuration of the environment ("quenched rwre"): a "Markov model" with Markov dependence in time, and a "quasi stationary" model with long range space–time correlations. We compare with the corresponding results for a model with i.i.d. (in space time) environment. In the range of times available to us the quenched distributions of the random walk displacement are far from gaussian, but as the behavior is similar for all three models one cannot exclude asymptotic gaussianity, which is proved for the model with i.i.d. environment. We also report results on the random drift and on some time correlations which show a clear power decay.

KEY WORDS: Random walks in random environment; computer simulations; Markov processes.

1. INTRODUCTION

In the late seventies and at the beginning of the eighties some remarkable papers by Solomon,[(1)] and Kesten *et al.*,[(2)] ending with the well known paper of Sinai,[(3)] gave a fairly complete picture of the behavior of discrete-time random walks on $\mathbb{Z}$ in a random environment which is i.i.d. in space

[1]Dipartimento di Matematica G. Castelnuovo, Università "La Sapienza", Piazzale Aldo Moro 2, 00185 Roma, Italy.
[2]Dipartimento di Matematica, Università di Camerino, via Madonna delle Carceri 9, 62032 Camerino, Italy.
[3]Dipartimento di Matematica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma, Italy; e-mail: pellegri@mat.uniroma3.it

and fixed in time. In particular for isotropic distribution Sinai showed that the random walk does not diffuse, and the displacement at time t is of the order of $\log^2 t$. It is not hard to see by heuristic arguments that such behavior is restricted to the one-dimensional case with fixed environment. In fact similar models on $\mathbb{Z}^\nu$ with fixed environment in dimension $\nu \geqslant 3$ diffuse if the random term is small enough, as shown by Bricmont and Kupiainen.[4] On the other hand, if the environment changes in time and is i.i.d. in space–time, diffusion also occurs in all cases in any dimension $\nu \geqslant 1$.[5]

One can ask what happens for random walks in an environment with space isotropic distribution and some kind of memory in time. There are very few rigorous results, and, except for case with an i.i.d. environment in space–time, mentioned above (which we will call "space–time independent model"), all results refer to Markov dependence in time.

As usual for such models one has to distinguish the "annealed" problem, which considers the random walk distribution induced by the environment, from the "quenched" one, in which the environment is fixed in space–time. If the environment is independent in space with a Markov dependence in time there are fairly complete results in all dimension $\nu \geqslant 1$ for the annealed problem, under some condition on the relaxation constant and the size of the random term.[6,7] Results for the corresponding quenched problem are available only in dimension $\nu \geqslant 3$.[8]

In the present paper we are mainly interested in the quenched problem for two models in dimension $\nu = 1$ with environment distributions that are independent in space with different behavior in time. The first model has a Markov dependence in time (to be called "Markov model"), and the other one (called "quasi-stationary", or "Q-S", model), has a long range dependence in time. We will also consider the space–time independent model mentioned above, for which much is known, for comparison.

The general model which we use for simulations is as follows. We consider some finite-range random-walk transition probabilities $\{P(x) : x \in \mathbb{Z}\}$, and a collection $\xi = \{\xi(t,x) : (t,x) \in \mathbb{Z}^2\}$ of random variables with $\xi(t,x) \in \mathcal{S}$ for some finite set $\mathcal{S}$. The environment distribution is a measure $\wp$ on $\Omega = \mathcal{S}^{\mathbb{Z}^2}$.

Taking a fixed configuration ξ of the environment, we consider the quenched random walk starting at the origin ($X_0 = 0$), with jump probabilities

$$P(X_{t+1} = x + u \mid X_t = x, \xi) = P(u) + c(u; \xi(t,x)), \tag{1a}$$

where $c(u; \cdot)$ is the random term, and is such that $\sum_u c(u; s) = 0$ for all $s \in \mathcal{S}$, and $\langle c(u; s) \rangle = 0$ for all $u \in \mathbb{Z}$, where $\langle \cdot \rangle$ denotes expectation with respect to the environment. We also need for consistency that $P(u) + c(u; s) \in [0, 1]$. The corresponding annealed problem refers to the distribution of the random walk displacement X_t induced by $\wp \times P$.

In the present paper, we only consider the following choices: P is the nearest neighbor symmetric random walk: $P(u) = 1/2$ if $|u| = 1$, and $P(u) = 0$ otherwise, $\mathcal{S} = \{\pm 1\}$, and for any $s \in \mathcal{S}$, $c(u; s) = (a/2)su$ if $u = \pm 1$ and $c(u; s) = 0$ otherwise, where $a \in (-1, 1)$ is a parameter. Hence

$$P(X_{t+1} = x + u | X_t = x, \xi) = \begin{cases} \frac{1}{2}(1 + au\xi(t, x)) & u = \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1b}$$

For the measure $\wp$ we will consider three choices.

(i) *The space–time independent model.* The variables $\{\xi(t, x) : (t, x) \in \mathbb{Z}^2\}$ are i.i.d. with the same distribution $\pi : \pi(\pm 1) = 1/2$.

(ii) *The Markov model.* For any fixed $x \in \mathbb{Z}$ the variables $\{\xi(t, x) : t = 0, 1, \dots\}$ are an independent copy of a Markov chain with some initial measure π_0 and transition matrix

$$Q = \begin{pmatrix} \epsilon & 1 - \epsilon \\ 1 - \epsilon & \epsilon \end{pmatrix} \tag{2}$$

depending on a parameter $\epsilon \in (0, 1)$.

(iii) *The quasi-stationary (or Q-S) model.* In this case $\xi(t, x) = \xi_1(t)\xi_2(x)$, where $\xi_1 = \{\xi_1(t) : t \in \mathbb{Z}\}$, $\xi_2 = \{\xi_2(x) : x \in \mathbb{Z}\}$ are two independent arrays of i.i.d. random variables with symmetric distribution $P(\xi_j(\cdot) = \pm 1) = 1/2,\ j = 1, 2$.

The space–time independent model (i) can be viewed as a special case of (ii) for $\epsilon = 1/2$. For $a = 0$ we get in all cases the usual simple nearest neighbor random walk, which we will call here "free" random walk, to stress that it does not feel the environment.

We will mainly be concerned with the quenched random walk. The annealed problem is trivial for the independent case, in the sense that the distribution of X_t induced by the product measure coincides with that of the free random walk $P^t =: \underbrace{P * P * \cdots * P}_{t \text{ times}}$, where $*$ denotes convolution. The same holds, as it is easy to see, for the Q-S model.

For the Markov case it is known from the results in refs. 6,7, and 9 that the annealed model has a diffusive behavior for any fixed $a \in (-1, 1)$

if the second eigenvalue of the matrix Q, $\mu = 1 - 2\epsilon$ is small enough, and for all $\epsilon \in (0, 1)$ if $|a| < 1/2$.

For the quenched problem it was proved that the space–time independent model is diffusive for $\wp$-almost all $\xi \in \Omega$, with the same parameters as for the free random walk.[5] However the model does not quite behave as the free random walk, as, e.g., there is a random drift $\mathcal{E}_t(\xi) = \mathbf{E}(X_t|\xi)$ (where $\mathbf{E}$ denotes expectation with respect to $\wp \times P$) which is of the order $t^{\frac{1}{4}}$ and is gaussian distributed as $t \to \infty$.[10] As shown in ref. 11, this fact implies that the first correction to the CLT asymptotics for the functionals $\sum_x f(\frac{x}{\sqrt{t}}) P(X_t = x|\xi)$ has anomalous large size.

No results are available for the quenched problems of the Markov and Q-S models. In both cases if there is some kind of "regular" behavior for the quenched random walk, it should be close to that of the annealed model. The main purpose of the present paper is to investigate such behavior by means of computer simulations. Simulations for the space–time independent models are used as a guideline for the interpretation of the results of the other models, in particular for what concerns the size of "asymptotic" times.

Here is a brief description of how computer simulations are performed. We start by generating a realization of $2L+1$ independent variables $\xi(0, x) : -L \leqslant x \leqslant L$ which take values ± 1 with probability $1/2$. The jump of X_t is determined with the probabilities (1b) depending on the field variable, and at each time the $2L+1$ variables are changed. The new variables are chosen independently of the previous ones in the independent case, and according to the Markov rules for Markov case. In the Q-S case the new variables are obtained by multiplying the previous ones by a factor ± 1, chosen with equal probability. If $X_t = \pm L$ before jumping, then 20 more positions $L < x \leqslant L + 10$ and $-L - 10 \leqslant x < -L$ are added, with corresponding variables $\xi(t, x)$.

The paper is organized as follows. In Section 2 we report results on the random drift $\mathcal{E}_t(\xi) = \mathbf{E}(X_t|\xi)$, which in all models has a size $\mathcal{O}(t^{\frac{1}{4}})$ and appears to be gaussian distributed. (For the space–time independent models this is proved in ref. 11.) In Section 3 we discuss on the basis of results of computer simulations whether the quenched distribution of X_t has a gaussian asymptotics for large t. Although the quenched dispersion is linear in time with good accuracy, the parameters of the χ^2 and Kolmogorov-Smirnov (K-S) tests stay very far away from reasonable values even at the largest times available for us (of the order of 10^6). The behavior is the same for all models, including the space–time independent model, for which asymptotic gaussianity is proved. In fact the values of the K-S test decrease with time, in the average, so that asymptotic gaussianity is not excluded for the quenched Markov and Q-S models.

Finally in Section 4 we report results on the decay of correlations of the jumps in time.

2. RANDOM DRIFT

In all three models the annealed drift vanishes by symmetry: $\mathbf{E}X_t = 0$. For the random (or "quenched") drift, setting $d(s) := \sum_u uc(u; s)$, and, taking into account that $\sum_u c(u; s) = 0$, we see that

$$\mathcal{E}_t(\xi) = \sum_x P(X_t = x|\xi)x = \mathcal{E}_{t-1}(\xi) + \sum_y P(X_{t-1} = y|\xi)d(\xi(t-1, y))$$

(where we omitted to specify the condition $X_0 = 0$), and iterating we finally find

$$\mathcal{E}_t(\xi) = \sum_{\tau=0}^{t-1} \sum_y P(X_\tau = y|\xi)d(\xi(\tau, y)). \tag{3}$$

If now the terms are orthogonal, i.e., $\langle P(X_\tau = y|\xi)d(\xi(\tau, y))P(X'_\tau = y'|\xi)d(\xi(\tau', y'))\rangle = 0$ for $(\tau, y) \neq (\tau', y')$, or approximately such, and the probabilities $P(X_\tau = y|\xi)$ are typically of the order $\tau^{-1/2}$, then it is easy to see that the L_2 norm of the random drift is of the order $t^{1/4}$. In fact for the space–time independent model it was proved in ref. 10 that, as $t \to \infty$, $t^{-1/4}\mathcal{E}_t(\xi)$ tends to a gaussian variable. For the other two models there are no results.

We present here results of computer simulations for only one choice of the parameters, and precisely we take $a = 0.7$ for all three models, with $\epsilon = 0.9$ for the Markov model. We generate by a random number generator $N = 1000$ independent copies of the environment ξ, up to time $T = 60,000$, and for each choice of ξ we compute at different values of $t \leqslant T$ the empirical average of X_t over $n = 10,000$ runs of the random walk.

Computer simulations (reported in Section 3) for all three models show that the quenched dispersion of X_t is bounded, in that range of times at least, by $\sigma^2 t$ with $\sigma^2 < 1.2$. (For model (i) there is, as we said, a rigorous result.) Hence, assuming, as it will turn out to be the case, that for models (ii) and (iii) the random drift is of the order of $t^{1/4}$, the empirical average will be close to the random drift if time t and the number of runs n are such that $\sqrt{t}/n$ is small. In our case this quantity does not exceed 0.025.

Figures 1a,b give the empirical variance of $\mathcal{E}_t(\xi)$ over the $N = 1000$ choices of ξ, for the Markov and the Q-S model, respectively. Figure 1c gives the same results for the space–time independent model. As we see in all three cases the variance behaves as $t^{1/2}$ with very good approximation, testified by the high values of R^2.

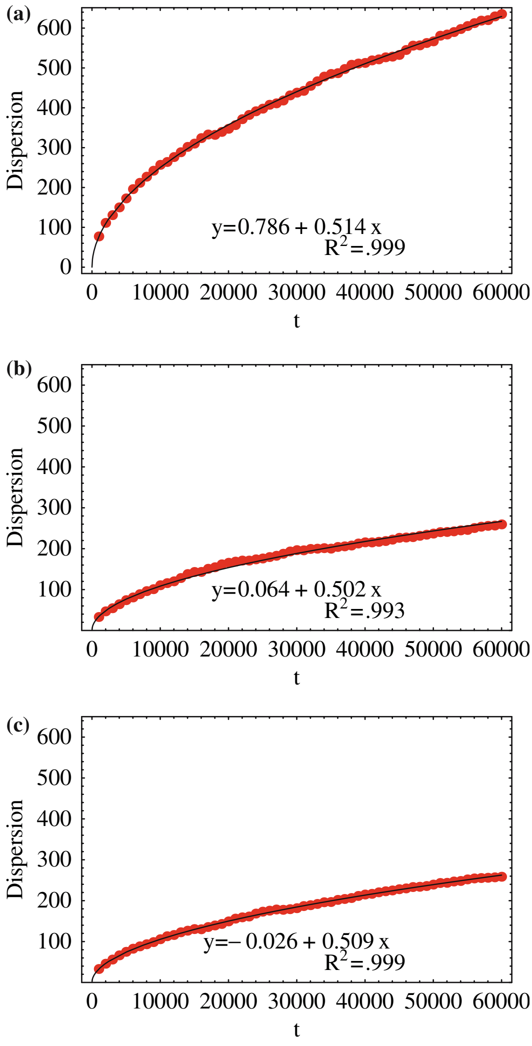

Fig. 1. Dispersion of $\mathcal{E}_t$ vs. t: (a) Markov Model; (b) Q-S Model; (c) Independent Model. (Formulas correspond to log–log plots.)

Table I. Test of Gaussianity for $\mathcal{E}_t$. $\chi^2_{0.95,38}$ Is the Quantil for 38 Degrees of Freedom with Confidence Level 95%, and $D_{1000;0.95}$ is the Quantil for the K-S Test with 1000 Data and the Same Confidence Level

Model	χ^2 test	$\chi^2_{0.95,38}$	K-S test	$D_{1000;0.95}$
Independent	38.44	53.38	0.02651531	0.043
Markov	27.56	53.38	0.01591915	0.043
Q-S	34.36	53.38	0.02062783	0.043

We tested the gaussian character of $\mathcal{E}_t$ for al three models at $t = 60,000$ by the K-S- and the χ^2-test, which are the standard tests in such cases (for details, see, e.g., ref. 12).

The K-S-test is perhaps not so well known as the χ^2-test, so we briefly define the quantil $D_{N;\alpha}$, for sample number N and confidence level $\alpha \in (0, 1)$. Let $S_N(x)$ be the distribution function of the empirical average of N independent samples of a random variable with distribution function $F(x)$. We consider a new random variable $D_N = \max_x |F(x) - S_N(x)|$, and define the quantil by setting $D_{N;\alpha} = x$ if $P(\{D_N < x\}) = \alpha$.

The results are reported in Table I. We see that the time we chose is large enough for the distribution to be gaussian with good accuracy in all three cases.

3. DO THE QUENCHED MODELS DIFFUSE?

One of the main open problems for the Markov and Q-S models is to establish whether the quenched distribution of X_t is diffusive or, maybe underdiffusive (it cannot be overdiffusive as the annealed models are diffusive).

An obvious remark in this respect is that if there is a typical asymptotic behavior of the quenched dispersion $D_t^2(\xi) =: \mathbf{E}((X_t - \mathcal{E}_t(\xi))^2|\xi)$ for large times, then by the previous considerations on the annealed model and our results on the random drift, it has to be linear in time. In fact the relation $\mathbf{E}(X_t^2) = \langle \mathcal{E}_t^2(\cdot)\rangle + \langle D_t^2(\cdot)\rangle$ implies that, as the annealed dispersion $\mathbf{E}(X_t^2)$ grows linearly in all three models, and, as we have seen $\langle \mathcal{E}_t^2(\cdot)\rangle \asymp t^{\frac{1}{2}}$, a typical asymptotic behavior of the quenched dispersion has to be linear in t.

Such behavior (which for the space–time independent model is a rigorous results) is supported by computer simulations. Figures 2a–c give the corresponding plots for the three models, for a fixed choice of ξ, in the

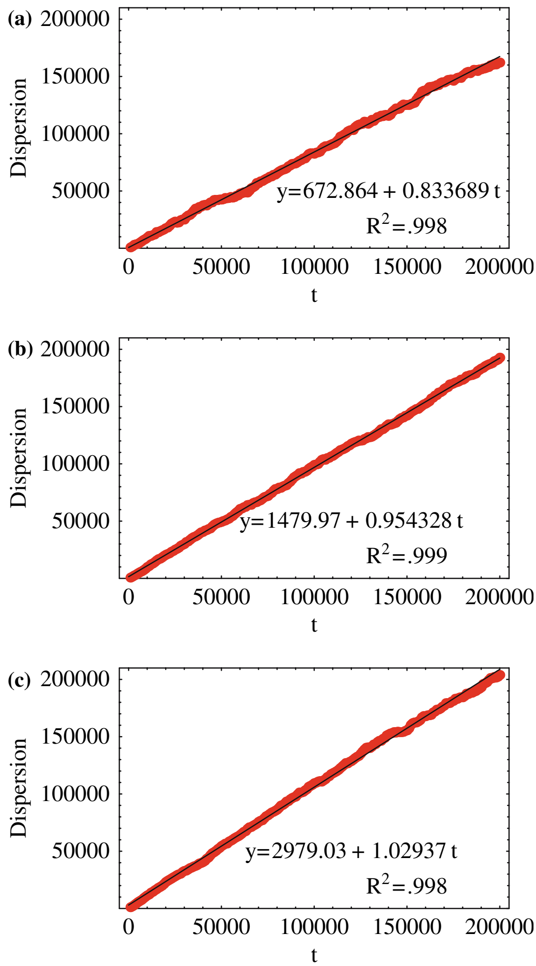

Fig. 2. (a) Dispersion vs. time: Markov Model. (b) Dispersion vs. time: Q-S Model. (c) Dispersion vs. time: Independent Model.

range up to $t = 200,000$. For each time the dispersion is taken over a sample of $n = 20,000$ independent runs of the random walk.

We have a linear growth in all three cases, with very good approximation, and picking different choices for the field ξ does not change the picture in significant way.

We have also applied direct tests of gaussianity for the quenched displacement X_t: the χ^2 and the K-S test. The values that we find are

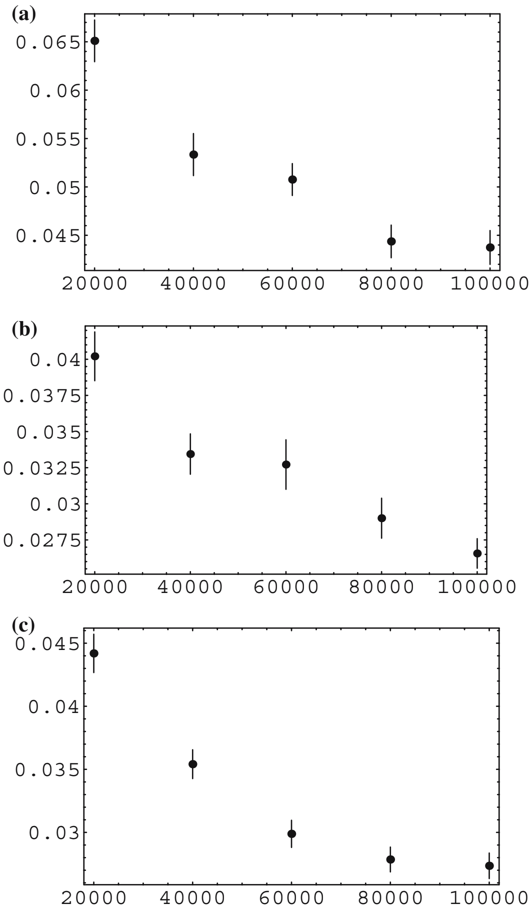

Fig. 3. D_{10^5} vs. time: (a) Markov Model; (b) Q-S Model; (c) Independent Model. (For reference: $D_{10^5;0.95} \approx 4 \cdot 10^{-3}$.)

however very far from what they should be, and the discrepancy persists even at times of the order 10^6, which taking into account that we need a large sample, are at the borderline of the reliability of our random number generator. The results of the tests seem however to improve very slowly as time grows. The trend is hardly detectable if we consider a single choice of ξ, as the results of the tests fluctuate heavily. But by taking averages of the results of the tests over several choices of ξ a slow monotonic fall off appears clearly, as shown for the K-S-test by Figs. 3a–c.

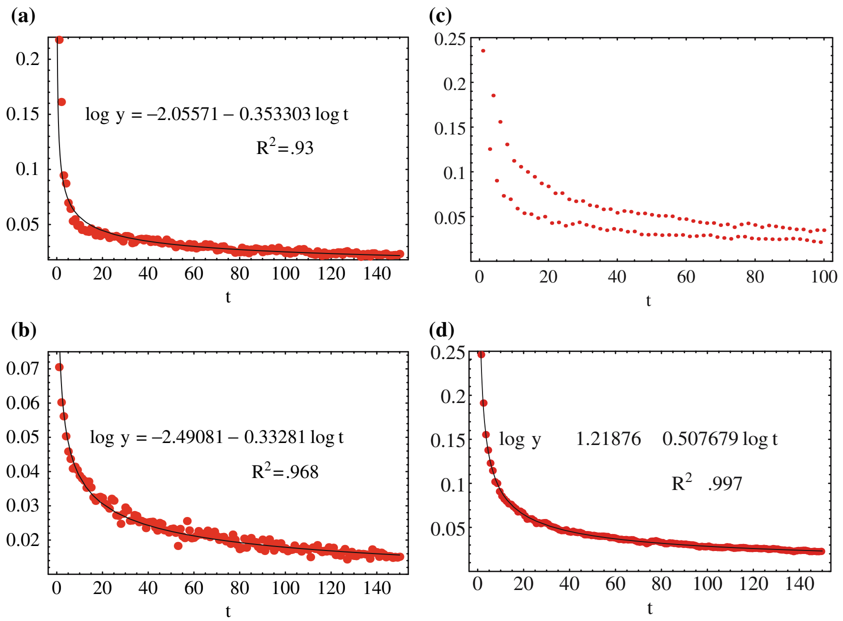

Fig. 4. (a) Quenched correlations vs. t: Markov Model. (b) Quenched correlations vs. t: Independent Model. (c) Quenched correlations vs. t: Q-S Model. (d) Quenched correlations vs. t: (Q-S Model with average).

The situation is more or less the same for all three models, including the space–time independent model, for which asymptotic gaussianity of the quenched distribution of X_t at large times, as we said above, is proved. It is perhaps not surprising that "asymptotic times" are so large, since the size of the random drift indicates that correction terms to the integral C.L.T. are of the order $t^{-\frac{1}{4}}$.

One can conclude that, in spite of the results of the K-S test (and of the χ^2 test, which are also very far from the proper range) there is no evidence to support the claim that possible limiting distributions for large times of the quenched random walks for the Markov and the Q-S models are non-gaussian.

4. BEHAVIOR OF THE QUENCHED TIME CORRELATIONS

We also considered the rate of decay in time of the quenched correlations of the increments. Setting $\Delta_t = X_{t+1} - X_t$ we define such quantities as

$$C_{t_1,t_2}(\xi) = \mathbf{E}(\Delta_{t_1}\Delta_{t_2}|\xi). \tag{4}$$

Again, for any single choice of ξ such quantities fluctuate heavily, and in order to find out a clear-cut pattern we take the average over 200 choices of the field. We computed the behavior of such averages as a function of the time difference $r = t_2 - t_1$ for a fixed value of $t_1 = 1000$, which appears to be large enough for a stable pattern.

The results are shown in Figs. 4a–c. It appears that the Q-S model has some kind of periodicity in time with period 2 and for a better comparison with the other models we took averages over a period (Fig. 4d).

The results show a fairly clear behavior, with a power decay very close to $t^{-1/3}$ for the Markov model and the space–time independent model and a faster decay, as $t^{-1/2}$ for the Q-S model.

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